An unconstrained binary model for the Uncapacitated Examination Timetabling Problem

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Abstract. Quantum computing is offering a novel perspective for solving combinatorial optimization problems. To explore the possibilities offered by quantum computers, the problems can be formulated as Quadratic Unconstrained Binary Optmization (QUBO) models, taking under consideration the limitations of the current state of Quantum Annealers. QUBO represents a class of optimization problems that involve binary decision variables and quadratic objective functions. It has applications in a wide range of fields and can be solved using classical or quantum optimization techniques, depending on the problem size and complexity. In this work, we provide a QUBO formulation of the Uncapacitated Examination Timetabling Problem along with modifications for symmetry reduction in the context of solving it on a quantum computer. We also introduce a test-bed dataset of small instances suitable for modern annealers, along with optimal solutions to serve for comparison. To prove the efficiency of the formulation we test our model in D-Wave's hybrid annealer.

Keywords: QUBO, Quantum Annealing, Scheduling, Hybrid Quantum Computing, UETP.

1 Introduction

Educational timetabling problems involve the task of scheduling courses, classes, examinations, teachers, and resources within an educational institution to optimize various objectives while satisfying constraints. These problems are common in schools, colleges, and universities, and they can be quite complex.

Examination timetabling is a critical administrative task in educational institutions that involves scheduling examinations for students, ensuring that all examinations are conducted smoothly, and minimizing conflicts or constraints. This process can be complex due to various factors, including room availability, student preferences, and the need to optimize resource utilization.

The UETP is a specific variant of the examination timetabling problem that focuses on scheduling a set of examinations within a given time frame and without considering room capacities or constraints related to room allocation. In other words, it assumes that all examinations can be accommodated in any available room, making it a simplified version of the more complex capacitated examination timetabling problem. UETP has practical applications in educational institutions where examinations need to be scheduled within a specific time frame without considering room constraints. It is a foundational problem in examination timetabling, and solutions to UETP can be further extended to handle capacitated versions of the problem.

While UETP does not consider room capacities, it still has constraints to satisfy:

- No two examinations for the same student should be scheduled in the same time slot (to avoid conflicts).
- Each examination can only be scheduled once.

To encourage greater preparation and less stress for the students, their schedules should also contain sufficient gaps between examinations for all students. Carter et al. [3] introduced in 1996 the problem along with a dataset made of real-life instances, and numerous researchers have experimented with this dataset since then.

Quantum annealing is a specialized quantum computing approach used to solve optimization problems. It is considered one of the quantum computing paradigms, alongside with other methods like quantum gate-based computing. Quantum annealers (QAs) are designed to tackle optimization problems by leveraging quantum properties to potentially find more efficient solutions than classical computers for specific types of problems. Quantum annealing and QUBO are closely related concepts in the field of quantum computing and optimization. QUBO is a mathematical formulation used to express certain optimization problems, and quantum annealing is a quantum computing approach that can be applied to solve QUBO problems.

An outline of the paper follows. Section 2 contains a glimpse of the broad bibliography regarding the UETP and education timetabling in general along with QUBO formulations of other scheduling problems. Section 3 provides a brief description of the problem along with symmetries that have been identified in the past. Section 4 introduces the dataset we created to allow instances of the UETP to fit in modern QAs. Section 5 contains the QUBO model accompanied with a minimal example. Finally, in Section 6 we demonstrate the results obtained by testing our dataset using D-Wave's [1] cloud-based hybrid solvers.

2 Related Work

The related work about the examination timetabling problem in general and UETP in particular is very large. We refer the interested readers to the survey papers [9] and [4] while our recent paper [5] uncovers some of the symmetries that are found in UETP.

Regarding QUBO a nice introduction to the subject can be found at [6]. More specifically, QUBO models have been tried for several scheduling and timetabling problems [11]. For example the nurse scheduling problem has been addressed using QUBO in [7]. Other examples can be found in [8], [12]. Another resource that is worth mentioning is [10] which presents a list of QUBO formulations for several optimization problems.

Quantum computing is a fascinating relatively new computing paradigm that holds the promise of surpassing the limits of computation that currently exist. It is based on a new non Von Neumann architecture and several technology companies invest large amounts of money and resources in an effort to realize such systems. D-Wave is a leading company for quantum computing and in this paper we use the so-called hybrid solver of D-Wave for our experiments. An evaluation of quantum and hybrid solvers for combinatorial problems can be found at [2] published on arXiv.

3 Problem Description

UETP instances contain students, the examinations they participate and the total number of available periods P for the entire timetable. Uncapacitated, as indicated, denotes the absence of room restrictions. Additionally, none of the other restrictions that are typical found in actual examination scheduling exist. These limitations include the availability of the examiners, the order of the examinations, the grouping of the times on weekdays, and others. Consequently, UETP can be seen as an abstraction of the actual examination scheduling problem.

An instance can be thought as an undirected weighted graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where vertices $\mathbb{V}$ represent examinations and edges $\mathbb{E}$ represent common students between examinations. The number of students who take both of the examinations at the edge's ends makes up the edge's weight W_{v_1,v_2}. The only strict requirements are that a) each examination should only be scheduled once, and b) no student should be permitted to take more than one examination per period. The quality of a timetable is measured by an objective function. Each student applies a penalty of 16, 8, 4, 2, 1 for intervals of 1, 2, 3, 4 or 5 periods between each of his examinations respectively. Notation used in this paper is shown in Table 1.

Table 1: Notation used for describing UETP.

Sets	
$\mathbb{V}$	Set of examinations.
$\mathbb{E}$	Set of pairs of examinations with students in common.
$\mathbb{P}$	Set of periods.
Constants	
F_{p_i,p_j}	$\begin{cases} 2^{5-\lvert p_i-p_j\rvert}, & \text{if } 0 < \lvert p_i - p_j\rvert \le 5 \\ 0, & \text{otherwise.} \end{cases}$
W_{v_1,v_2}	Total number of common students between examinations v_1 and v_2.

$$x_{v,p} = \begin{cases} 1, \text{ if examination } v \text{ is placed in period } p. \\ 0, \qquad \text{otherwise.} \end{cases} \quad \forall v \in \mathbb{V} \quad \forall p \in \mathbb{P} \tag{1}$$

$$\min \sum_{(v_1,v_2)\in\mathbb{E}} \sum_{p_1\in\mathbb{P}} \sum_{p_2\in\mathbb{P}} F_{p_1,p_2} W_{v_1,v_2} x_{v_1,p_1} x_{v_2,p_2} \tag{2}$$

$$\text{s.t.} \quad x_{v_1,p} + p_{v_2,p} <= 1 \quad \forall (v_1, v_2) \in \mathbb{E} \quad \forall p \in 1..P \tag{3}$$

$$\sum_{p=1}^{P} x_{v,p} = 1 \quad \forall v \in \mathbb{V} \tag{4}$$

The penalty factor for the intervals of periods is calculated as in Table 1. Binary decision variables in Equation 1 denote the period that each examination is placed to. The objective function in equation 2 simply totals the penalties for all examinations. Finally, constraint 3 ensures than no two examinations sharing students are in the same period and constraint 4 obligates each examination to be placed once and only once.

3.1 Bidirectional Timetable Symmetry

It is easy to observe that if period $p \in 1..P$, changes to $P-p+1$ for all examinations, then we effectively get the original solution reversed. Since the cost is computed based on the distance among periods of scheduled examinations, the objective function is unaffected as demonstrated in equation 5.

$$\sum_{(v_1,v_2)\in\mathbb{E}} \sum_{p_1\in\mathbb{P}} \sum_{p_2\in\mathbb{P}} F_{p_1,p_2} W_{v_1,v_2} x_{v_1,p_1} x_{v_2,p_2} = \sum_{(v_1,v_2)\in\mathbb{E}} \sum_{p_1\in\mathbb{P}} \sum_{p_2\in\mathbb{P}} F_{p_1,p_2} W_{v_1,v_2} x_{v_1,(P-p_1+1)} x_{v_2,(P-p_2+1)} \tag{5}$$

4 Dataset

Different datasets regarding the UETP problem were made public over the years, but the sheer size of the included instances make them to big to fit in current state of the art annealers. While the number of qubits required for some small instances is acceptable, the nature of the problem i.e., the relation of two exams with students in common, results in an increase of the Non Zero Couplings in the matrix that is sent to the solver (usually called a QMatrix) provided to the solver, thus making most of these instances unfit for the annealer.

In order to demonstrate the proof of concept we opted to generate a dataset consisting of 50 small instances able to run on current annealers. To create an instance we randomly choose between 3 and 7 exams and generate a complete graph with them (all of them have students in common) the number of the periods available equals the number of nodes in the complete graph to make the instance compact e.g., there exists no solution with an empty period, then we proceed to add more exams and more conflicts while keeping the number of conflicts under 60. The students in common between the conflicting exams (the weight of their edge) is chosen arbitrarily between 1 and 100. However, this number could be higher as this will not result in more variables.

To test the annealer against the optimal solutions we employ GoogleOR-Tools CP-SAT Solver to solve the problem instances to optimality. The characteristics and optimal solutions values are presented in Table 2.

Table 2: Instances and characteristics

Instance	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Examinations	20	19	8	13	22	19	20	17	18	20
Periods	6	6	5	5	4	6	7	5	7	6
Conflict density	0.23	0.28	0.37	0.55	0.19	0.25	0.24	0.3	0.27	0.27
Optimal	10512	10736	13540	17376	16152	9916	7458	13242	8365	12720
Instance	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Examinations	24	24	20	18	19	15	23	17	27	8
Periods	4	4	6	5	4	5	5	5	7	6
Conflict density	0.15	0.15	0.23	0.29	0.25	0.38	0.17	0.33	0.12	0.38
Optimal	16312	13460	10018	13088	16700	12724	9640	11704	8281	12255
Instance	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
Examinations	19	19	26	17	20	19	27	23	19	17
Periods	4	5	7	7	6	7	7	7	6	4
Conflict density	0.25	0.27	0.14	0.29	0.24	0.24	0.13	0.17	0.24	0.32
Optimal	15592	15628	5316	7464	10445	8301	5718	9265	7700	19680
Instance	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
Examinations	21	25	17	15	22	26	20	17	18	19
Periods	6	6	7	4	4	7	6	5	5	7
Conflict density	0.21	0.16	0.33	0.39	0.19	0.14	0.24	0.31	0.29	0.25
Optimal	8749	8834	8780	20364	17412	6735	11691	12356	14434	7221
Instance	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
Examinations	24	15	17	21	23	8	18	24	19	18
Periods	7	5	5	4	6	6	5	6	6	6
Conflict density	0.16	0.44	0.33	0.23	0.17	0.38	0.27	0.18	0.25	0.3
Optimal	5885	15108	18630	24604	8456	9731	7964	8723	9407	9654

5 Unconstrained Binary Model

The general form of a QUBO objective function can be expressed as follows:

$$\min \quad \sum_{i=1}^{n} q_{ii} x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} q_{ij} x_i x_j$$

where:

- x_i are binary variables.
- n is the number of binary variables.
- q_{ii} represents the linear coefficient associated with variable x_i.
- q_{ij} represents the quadratic coefficient associated with the interaction between variables x_i and x_j.

The model for a QUBO problem will always be the same. What makes the difference is the choice of the values in the QMatrix. For this problem our binary decision variables assume the value 1 when a specific exam is scheduled in a period. For exams in conflict the corresponding quadratic coefficient is calculated as $F_{p_1,p_2} W_{v_1,v_2}/2$. We divide by two because the QMatrix is symmetric. As the nature of QUBO formulation is inherently unconstrained we choose a large enough number M to impose penalties and incentives in the objective function that can act as constraints. We chose M to equal the sum of all edges multiplied by 16 to ensure that no worse solution exists when you violate the conflicting exams constraint 3. To provide the incentive to schedule all exams, as dictated by constraint 4, we set the value of an exam being placed to $-M$ and to M if the exam is placed twice.

We use a minimal problem presented in Figure 1 to demonstrate the resulting QMatrix in Table 3. Note that this toy example involves 5 examinations and 3 periods.

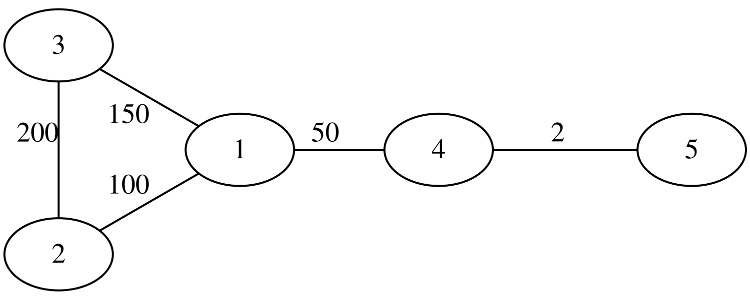

Fig. 1: Minimal problem graph (5 examinations, 3 periods).

We can also choose to eliminate the bidirectional timetable symmetry discussed in Section 3.1 by restricting any two conflicting exams to be placed in a certain order. If

Table 3: Q Matrix

	E_1P_1	E_1P_2	E_1P_3	E_2P_1	E_2P_2	E_2P_3	E_3P_1	E_3P_2	E_3P_3	E_4P_1	E_4P_2	E_4P_3	E_5P_1	E_5P_2	E_5P_3
E_1P_1	-M	M	M	M	800	400	M	1200	1200	M	400	200	0	0	0
E_1P_2	M	-M	M	800	M	800	1200	M	1200	400	M	400	0	0	0
E_1P_3	M	M	-M	400	800	M	1200	1200	M	200	400	M	0	0	0
E_2P_1	M	800	400	-M	M	M	M	1600	800	0	0	0	0	0	0
E_2P_2	800	M	800	M	-M	M	1600	M	1600	0	0	0	0	0	0
E_2P_3	400	800	M	M	M	-M	800	1600	M	0	0	0	0	0	0
E_3P_1	M	1200	1200	M	1600	800	-M	M	M	0	0	0	0	0	0
E_3P_2	1200	M	1200	1600	M	1600	M	-M	M	0	0	0	0	0	0
E_3P_3	1200	1200	M	800	1600	M	M	M	-M	0	0	0	0	0	0
E_4P_1	M	400	200	0	0	0	0	0	0	-M	M	M	M	16	8
E_4P_2	400	M	400	0	0	0	0	0	0	M	-M	M	16	M	16
E_4P_3	200	400	M	0	0	0	0	0	0	M	M	-M	8	16	M
E_5P_1	0	0	0	0	0	0	0	0	0	M	16	8	-M	M	M
E_5P_2	0	0	0	0	0	0	0	0	0	16	M	16	M	-M	M
E_5P_3	0	0	0	0	0	0	0	0	0	8	16	M	M	M	-M

we choose that exam 1 must be scheduled before exam 3 the binary value E_1P_1 is not needed anymore as exam 1 cannot be placed in the first period and for each combination where exam 1 is scheduled before exam 3 we again provide the value of M to place a heavy penalty if such a combination is selected. The QMatrix with these modifications is presented in Table 4.

Table 4: Q Matrix without bidirectional timetable symmetry

	E_1P_2	E_1P_3	E_2P_1	E_2P_2	E_2P_3	E_3P_1	E_3P_2	E_3P_3	E_4P_1	E_4P_2	E_4P_3	E_5P_1	E_5P_2	E_5P_3
E_1P_2	-M	M	800	M	800	1200	M	**M**	400	M	400	0	0	0
E_1P_3	M	-M	400	800	M	1200	1200	M	200	400	M	0	0	0
E_2P_1	800	400	-M	M	M	M	1600	800	0	0	0	0	0	0
E_2P_2	M	800	M	-M	M	1600	M	1600	0	0	0	0	0	0
E_2P_3	800	M	M	M	-M	800	1600	M	0	0	0	0	0	0
E_3P_1	1200	1200	M	1600	800	-M	M	M	0	0	0	0	0	0
E_3P_2	M	1200	1600	M	1600	M	-M	M	0	0	0	0	0	0
E_3P_3	**M**	M	800	1600	M	M	M	-M	0	0	0	0	0	0
E_4P_1	400	200	0	0	0	0	0	0	-M	M	M	M	16	8
E_4P_2	M	400	0	0	0	0	0	0	M	-M	M	16	M	16
E_4P_3	400	M	0	0	0	0	0	0	M	M	-M	8	16	M
E_5P_1	0	0	0	0	0	0	0	0	M	16	8	-M	M	M
E_5P_2	0	0	0	0	0	0	0	0	16	M	16	M	-M	M
E_5P_3	0	0	0	0	0	0	0	0	8	16	M	M	M	-M

6 Experiments and results

Our experiments were performed using the hybrid Quantum Annealer provided by D-Wave. A time limit of 20 seconds was given for each problem instance and the results are presented in Table 5. The justification for using only 20 seconds of running time per instance is due to the small sizes of the problems and the limited time that the hybrid solver of D-Wave can use the Quantum infrastructure for the non-pay version of D-Wave Leap.

Results show that there is potential for using Quantum Annealers for solving UETP problems. Some results are optimal, while others are near optimal, as can be seen in Figure 2 which shows how far from the optimal solution the results for the 50 problem instances are.

Table 5: Results

Instance	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Decision Variables	119	113	79	64	87	113	139	84	125	119
Non Zero Couplings	1950	2061	1290	1215	908	1851	2596	1230	2370	2226
Objective	12082	12687	13540	17376	16784	11265	9515	13302	9176	14899
Difference	3.47%	3.26%	11.38%	0.00%	6.62%	0.60%	6.58%	4.12%	3.56%	4.16%
Instance	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Decision Variables	95	95	119	89	75	74	114	84	188	95
Non Zero Couplings	864	864	1950	1310	842	1165	1405	1330	2669	1926
Objective	17504	15336	11244	13996	16700	13312	11346	12630	11224	12255
Difference	0.00%	0.00%	0.96%	3.18%	6.06%	0.11%	2.31%	3.94%	1.76%	2.88%
Instance	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
Decision Variables	75	94	181	118	119	132	188	160	113	67
Non Zero Couplings	826	1375	2729	2277	2004	2392	2904	2645	1821	814
Objective	15592	16662	8448	8584	12264	9922	9499	11914	9091	19680
Difference	1.68%	0.00%	1.13%	4.06%	1.90%	7.54%	0.00%	0.00%	1.60%	3.49%
Instance	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
Decision Variables	125	149	118	59	87	181	119	84	89	132
Non Zero Couplings	1971	2229	2524	770	880	2788	2016	1250	1325	2439
Objective	10426	11357	10342	20364	17968	10187	12908	12564	14794	8798
Difference	4.01%	4.45%	12.42%	6.25%	4.14%	0.00%	4.37%	6.25%	4.08%	0.79%
Instance	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
Decision Variables	167	74	84	83	137	95	89	143	113	107
Non Zero Couplings	2638	1320	1335	946	1977	1890	1250	2214	1857	1974
Objective	8789	15108	18790	24604	11036	10404	8156	11369	11097	11133
Difference	10.20%	2.47%	0.42%	0.62%	4.92%	9.90%	0.00%	0.21%	0.00%	1.67%

7 Conclusions

In this paper we tried to present a proof of concept idea about using a QUBO formulation and a Quantum Annealer solver for solving UETP. We created a custom dataset of small problem instances keeping in mind the current limitations of the Quantum Annealers and modeled the problem according to QUBO. We then run experiments on the non-pay version of D-Wave's Leap architecture. Our results are promising, although they do not manage to find optimal solutions for all problem instances given only 20 seconds for each problem, they achieve near optimal results for most of the cases.

Fig. 2: Difference in percentage from the optimal solution for 50 problem instances.

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